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GROWTH OF THE SUDLER PRODUCT OF SINES AT THE GOLDEN ROTATION NUMBER

PAUL VERSCHUEREN[†] AND BEN MESTEL^{*}

ABSTRACT. We study the growth at the golden rotation number $\omega = (\sqrt{5} - 1)/2$ of the function sequence $P_n(\omega) = \prod_{r=1}^n |2 \sin \pi r \omega|$. This sequence has been variously studied elsewhere as a skew product of sines, Birkhoff sum, q-Pochhammer symbol (on the unit circle), and restricted Euler function. In particular we study the Fibonacci decimation of the sequence P_n , namely the sub-sequence $Q_n = \left| \prod_{r=1}^{F_n} 2 \sin \pi r \omega \right|$ for Fibonacci numbers F_n , and prove that this renormalisation subsequence converges to a constant. From this we show rigorously that the growth of $P_n(\omega)$ is bounded by power laws. This provides the theoretical basis to explain recent experimental results reported by Knill and Tangeman (Self-similarity and growth in Birkhoff sums for the golden rotation. *Nonlinearity*, 24(11):3115–3127, 2011).

Keywords: Asymptotic growth, renormalisation, self-similarity, sine product

AMS 2010: 37D45, 26A12, 37E40, 41A60, 60G18

1. INTRODUCTION

We will study the sequence of positive real functions $P_n(\omega)$ which we define as follows:

$$(1.1) \quad P_n(\omega) = \prod_{r=1}^n |2 \sin \pi r \omega|$$

This function sequence arises in a surprising number of fields of pure and applied mathematics and physics, disguised by a number of different representations and terminologies. In pure mathematics there are important applications in partition theory ([1, 2]), Padé approximation and continued fractions (see [3] for a list of 13 examples), whereas in applied mathematics the function arises in the study of strange non-chaotic attractors (SNA's), KAM theory, and q-series which is heavily used in string theory (see [4, 5, 6, 7, 8]). For example, in the context of SNA's (our own area of interest) the functions (1.1) arise in the renormalisation analysis of skew products. Figure 1.1 shows renormalised graphs of the function for various values of n over the Fibonacci interval $[1..F_n]$. Note the self-similarity, and also the approximately

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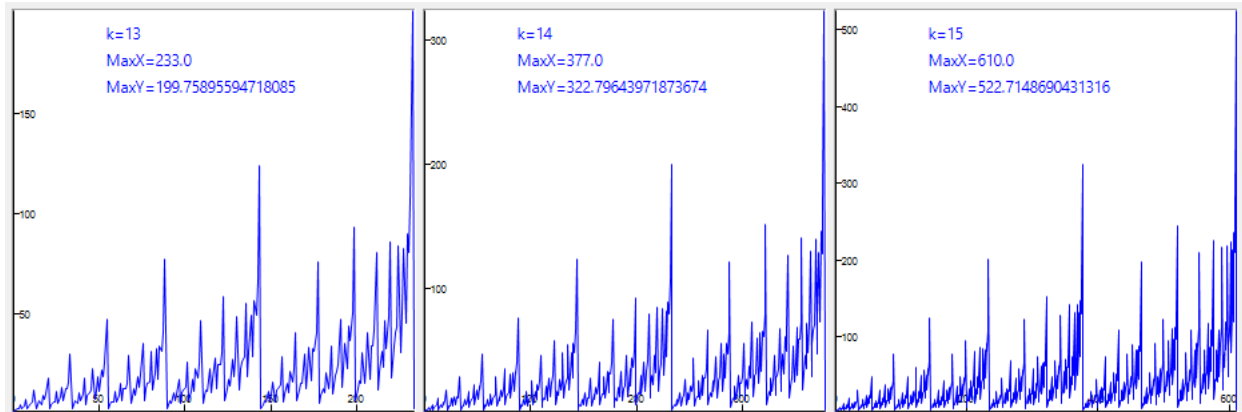


FIGURE 1.1. Renormalised graphs of $P_n(\omega)$ against n in the range $1..F_k$. Notable features are the self-similarity of the function and the almost linear growth of the peaks. Note also that the peaks are substantially higher than other values of the function.

linear growth of the peaks. The common problem in each of these areas is to understand some aspect of the growth of the sequence.

This list of application areas is no doubt incomplete if only because of the remarkable range of representations and terminology under which the function sequence appears. The representation $P_n(\omega) = \prod_{r=1}^n |2 \sin \pi r \omega|$ arises in dynamical systems as the magnitude $|z|$ in the skew product $(\theta, z) \mapsto (\theta + \omega, 2z \sin \pi \theta)$ (with initial condition $(\omega, 1)$). However putting $z = \exp(2i\pi\omega)$ we obtain the representation $P_n(\omega) = \prod_{r=1}^n |1 - z^r|$ which is the modulus of the restricted Euler function, and links us to partition theory (amongst other things). Further if we take the q -Pochhammer symbol $(a; q)_n$ and put $a = q = z = \exp(2i\pi\omega)$ we have $P_n(\omega) = |(z; z)_n|$ which links us to q -series and string theory. Finally we have $\log P_n(\omega) = \sum_1^n f(r\omega)$, where $f(x) = \frac{1}{2} \log(2 - 2 \cos 2\pi x)$. The latter function is the harmonic conjugate of the sawtooth function $\pi(\{x\} - \frac{1}{2})$ (see [9]). The sum $\sum_1^n f(r\omega)$ is the Birkhoff sum of f , and links us to ergodic theory and KAM theory.

2. A BRIEF SURVEY OF KEY RESULTS

It seems that the function $P_n(\omega)$ has to date been studied relatively independently in at least several of the fields noted above, no doubt partly as a result of the diverse terminology and representations in play. It seems worthwhile drawing together the various strands of study, and we provide a brief survey below.

2.1. Initial remarks. The situation is very simple if ω is rational. Let $\omega = p/q$ where p/q is in its lowest terms, then $P_n(\omega) = 0$ for $n \geq q$. In addition there is an old and rather elegant result we shall make heavy use of, namely that $\prod_{r=1}^{q-1} 2 \sin \pi r p/q = q$, which becomes in our notation $P_{q-1}(p/q) = q$.

When ω is irrational, $P_n(\omega)$ is never 0. The contrast with the case of ω rational suggests that the number theoretic properties of ω are important, which is indeed confirmed by later results. In addition, if for some ω_0 we have $\limsup P_n(\omega_0) > 0$, then since $\lim P_n(\omega) = 0$ for rational ω arbitrarily close to ω_0 , the behaviour of the sequences $P_n(\omega)$ are highly sensitive to the value of ω around ω_0 - a harbinger of chaotic behaviour in any Dynamical Systems incorporating such orbits.

2.2. Growth with n of the norm $\|P_n(\omega)\| = \sup_{\omega} |P_n(\omega)|$. The first in-depth study of the function $P_n(\omega)$ seems to have been made by Sudler in 1964[1], although Erdős & Szekeres previously stated a “very easy” result (without proof) in 1959[10]¹. Sudler² showed that in the limit the norm grows exponentially with n ,³ with the growth rate E being given by the formula:

$$E = \lim_{n \rightarrow \infty} \|P_n(\omega)\|^{1/n} = \omega_0^{-1} \int_0^{\omega_0} \log |2 \sin \pi \omega| d\omega = 1.2197...$$

where ω_0 is the (unique) solution in $[1/2, 1]$ of $\int_0^{\omega_0} \omega \cot \pi \omega d\omega = 0$. Further he also showed that $\|P_n(\omega)\|$ is achieved at ω_n where $\omega_n \sim \omega_0/n \in [1/2n, 1/n]$ as n grows.

Freiman and Halberstam (1988) [11] later provided an alternate proof which gives the same result in the even more elegant form $E = 2 \sin \pi \omega_0$ (where ω_0 is as above). (Incidentally this seems to be the first paper to treat the function $P_n(\omega)$ as a first class citizen, ie as worthy of study in its own right rather than as a stepping stone to the estimation of other functions).

In 1998 Bell et al [12] proved a number of stronger results, in particular that the norm of the decimated sub-product $\|\prod_1^n 2 \sin r^k \omega\|$ grows exponentially for any $k \geq 1$. More recently Jordan Bell (2013) [13] adapted the method of Wright [2] to show $\|P_n(\omega)\| \sim C_1 \sqrt{n} E^n$, and also generalised the result to the L_p norm: $\|P_n(\omega)\|_p = \left(\int_0^1 P_n(\omega)^p d\omega \right)^{1/p} \sim C_1 (C_2 n^{-3/2})^{1/p} \sqrt{n} E^n$ for calculated constants C_1, C_2 .⁴

¹Their claim was that $\lim_{n \rightarrow \infty} \|P_n(\omega)\|^{1/n}$ exists and lies between 1 and 2. Sudler found the limit precisely.

²Freiman and Halberstam (1988) attribute this result to Wright[11] but from a careful reading of both papers [1, 2] it seems that Sudler has priority. Sudler does however acknowledge the help of Wright as a referee in improving the proofs, and Wright also improves Sudler's result in his own subsequent paper.

³More precisely he showed $\|P_n(\omega)\|^{1/n} = E + O(\log n/n)$ where E is the constant above

⁴ C_2 is actually $O(p^{-1/2})$, but is independent of n .

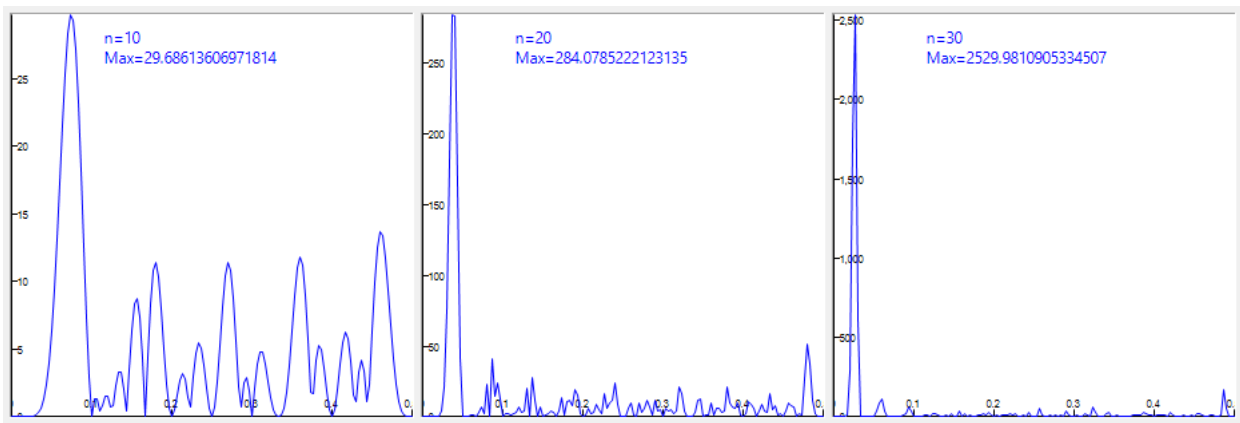


FIGURE 2.1. $P_n(\omega)$ plotted over the interval $\omega \in [0.0, 0.5]$ for various values of n . Note how the norm is achieved at a point which is converging on the origin, but that the exponential growth achieved here is uncharacteristic.

2.3. Growth of peaks of the sequence $P_n(\omega)$ at fixed ω . We might expect that as the norm of the function $P_n(\omega)$ grows exponentially, then the pointwise growth rate (the growth rate of the sequence $P_n = P_n(\omega)$ at a fixed value of ω) would also be exponential. However this turns out not to be the case. Using the theory of uniform distribution, Lubinsky [14] showed that for almost all ω , $\lim_{n \rightarrow \infty} P_n^{1/n} = 1$, ie the growth is at best sub-exponential, not exponential. This apparent conflict is explained by Figure 2.1 in which we see that the exponential growth of the norm is achieved at a peak which is uncharacteristic of the rest of the function. This peak narrows and converges on 0 as n grows, so that for any fixed value of ω the peak will pass it for some value of n , after which the growth at that point will revert to being sub-exponential.

In [9] Knill & Lesieutre adapted Herman's Denjoy-Koksma result [15] to show for some constant C that $P_n(\omega) < n^{Cn^{1-1/s} \log n}$ when ω is of Diophantine type $s \geq 1$.⁵

Lubinsky [3] studied the problem in the context of q -series and showed that, for almost all ω , and all $\epsilon > 0$, there are constants N, C (dependent on ω, ϵ) such that for all $n > N$ we have $P_n(\omega) \leq n^{C(\log n)^\epsilon}$, and further, when ω has bounded partial quotients, that $P_n(\omega) \leq n^C$. Amongst other results he also showed that, for all irrational ω , $\limsup_n P_n(\omega) \geq n$ (from which we deduce $C \geq 1$ above), and conjectured that for all ω , $\liminf P_n(\omega) = 0$. He established that the latter result certainly holds for ω with unbounded partial quotients.

2.4. Growth of the sequence $P_n(\omega)$ when ω is the golden rotation $(\sqrt{5} - 1)/2$. The results of the previous section bound the peak growth of $P_n(\omega)$. However the peaks of this function are very different from its values elsewhere (see Figure 1.1). Certain applications, in particular in the study of strange non-chaotic attractors (SNA's), require a sharper estimate of the size of $P_n(\omega)$ at every point n , and not just at the peaks.

Working in the context of KAM theory, Knill and Tangerman studied the Birkhoff sum representation $S_n(\omega) = \sum_{r=1}^n \log(2 - 2 \cos 2\pi r\omega)$ in their 2011 paper [6]. It is easy to show that $S_n(\omega) = 2 \log P_n(\omega)$. As is often the case with this type of problem, they chose to study the "simplest" irrational number, the golden mean (or more accurately, its fractional part $\omega = (\sqrt{5} - 1)/2$) which is of Diophantine type 1 and has rational convergents F_{n-1}/F_n where F_n is the n th Fibonacci number (indexed from $F_0 = 0$). Taking the sequence (F_n) as a renormalisation scale, they presented experimental graphical and numerical evidence for the existence of an asymptotic renormalisation function. The renormalisation approach was also earlier studied by Kuznetsov et al (1995) [5] in a slightly more general setting, where they used polynomial approximation to obtain strong numerical evidence also for asymptotic renormalisation functions.

Assuming the existence of this asymptotic function as a hypothesis, Knill and Tangerman deduced the following consequences:

Theorem 2.1. *Consequences of the hypothesis*

⁵ie There is some constant $c > 0$ such that $|\omega - p/q| > c/q^{1+s}$ for any rational p/q in lowest terms

- (1) The log average of the Birkhoff sum tends to a constant along the renormalisation subsequence, ie $S_{F_n}(\omega)/\log F_n \rightarrow c$ for some constant c .
- (2) The sequence $S_n(\omega)/\log n$ has accumulation points at 0 and 2.
- (3) The sequence $S_n(\omega)/\log n$ is bounded.⁶

In an archive paper [8] (2012) Knill studied a related sum of cotangents $S'_n(\omega) = \sum_{r=1}^n \cot \pi r \omega$, and demonstrated that this sum has marked self-similarity. From this he also sketched how one might derive the results above. In this paper we will give a detailed proof of the slightly stronger results which are set out below in terms of $P_n(\omega)$. But $S_n(\omega) = \sum_{r=1}^n \log(4 \sin^2 2\pi r \omega) = 2 \log P_n(\omega)$ and from this it is easily seen that the results below imply the results above. (Note that (2) above is the result of combining (1),(2) below). In addition the approach is complementary to that of Knill.

Theorem 2.2. *The following results hold for the golden rotation number ω :*

- (1) For some constant c , $P_{F_n}(\omega) \rightarrow c$.
- (2) For the same constant c , $P_{F_n-1}(\omega)/F_n \rightarrow c/2\pi\sqrt{5}$.
- (3) There are real constants $C_1 \leq 0 < 1 \leq C_2$ such that $n^{C_1} \leq P_n(\omega) \leq n^{C_2}$.

The proof of the first foundational result, namely that $P_{F_n}(\omega) \rightarrow c$ for some constant c , will occupy the bulk of the paper. In section 8 we will deduce results (2) and (3).

3. STATEMENT OF MAIN RESULT & OVERVIEW OF PROOF

At the end of the previous section we described how Knill & Tangerman recently presented experimental graphical and numerical evidence for the existence of an asymptotic renormalisation function when $\omega = (\sqrt{5} - 1)/2$. From this they deduced three consequences. However we will show in Section 8 that the second and third consequences flow directly from the first, and have no dependency on the experimental function. Our main contribution in this paper is to establish the first consequence rigorously without reference to the experimental function, that is, we will prove that the sequence $P_{F_n}(\omega)$ converges to a constant.

This rather simple statement belies the surprising amount of work which seems necessary to prove it. However it is worth noting that both Knill and Lubinsky remark that this is one problem area where established procedures and powerful tools fall short. This has also been our own experience, and we have felt very much forced back to a proof from first principles.

Following renormalisation terminology, we “decimate” the sequence $P_n = |\prod_{r=1}^n 2 \sin \pi r \omega|$ by picking every F_n th element to yield a “renormalisation sub-sequence” $Q_n = \left| \prod_{r=1}^{F_n} 2 \sin \pi r \omega \right|$. Our main result is now the following:

Theorem 3.1. *The sequence $Q_n = \left| \prod_{r=1}^{F_n} 2 \sin \pi r \omega \right|$ is convergent to a constant $c = 2.407\dots$*

The proof of this theorem will occupy the main body of the paper (sections 3–7).

In Section 8 we deduce from the main result two other results reported by Knill & Tangerman. In particular this includes the result that the Sudler product growth at ω is bounded by a power law. Knill & Tangerman suggested that this particular result would flow from a modification of the proof of the Denjoy-Koksma result in ergodic theory, but on closer examination further work appears necessary. We have again found the need to derive this corollary from first principles.

⁶Lubinsky [3] proved this result for almost all ω (in the q -Pochhammer form $\log |(q; q)_n| = O(\log n)$), but not necessarily for the specific value of the golden mean

3.1. Overview of the proof of the main result (sections 4–7): In section 4, together with some other preliminaries, we separate out the proofs of a number of ancillary results from the overall flow, in an attempt to make clearer the main lines of reasoning in the other sections.

In section 5 we introduce a core strategy which is to exploit the continued fraction convergents to the inverse golden mean ω . These convergents are the ratios of subsequent Fibonacci numbers F_{n-1}/F_n , and $\omega = (F_{n-1} - (-\omega)^n)/F_n$ (see (4.8)). This gives us:

$$Q_n = \left| \prod_{r=1}^{F_n} 2 \sin \pi r \omega \right| = \left| \prod_{r=1}^{F_n} 2 \sin \pi r (F_{n-1} - (-\omega)^n) / F_n \right|$$

This allows us to develop a representation of Q_n as a product of three rather more tractable products, namely:

$$Q_n = A_n B_n C_n = (2F_n \sin \pi \omega^n) \left(\prod_{t=1}^{F_{n-1}} \frac{s_{nt}}{2 \sin \pi \frac{t}{F_n}} \right) \prod_{t=1}^{F_n/2} \left(1 - \frac{s_{n0}^2}{s_{nt}^2} \right)$$

where $s_{nt} = 2 \sin \pi (t/F_n - \omega^n ([F_{n-1}t]/F_n - 1/2))$.

It is easy to show that $A_n \rightarrow 2\pi/\sqrt{5}$, and in fact the products B_n, C_n also converge to strictly positive limits. However the latter demonstrations require significantly greater effort, and receive their own sections.

In section 6 we shall deal with the convergence of the simpler of the two products, namely $C_n = \prod_{t=1}^{F_n/2} \left(1 - \frac{s_{n0}^2}{s_{nt}^2} \right)$. In section 7 we shall deal with the convergence of $B_n = \left(\prod_{t=1}^{F_{n-1}} \frac{s_{nt}}{2 \sin \pi \frac{t}{F_n}} \right)$. This requires the most work and is broken down into several significant sub-sections.

4. PRELIMINARIES

4.1. Notation. We will make use of both modulo arithmetic and floor functions. Since the box notation $[.]$ is often used for both purposes, in this paper we will use the following conventions:

- For a given positive modulus $q \geq 1$, we use $[r]$ to represent the residue of $r \bmod q$ in the residue set $\{0, \dots, q-1\}$, for example $[-1] = q-1$.
- We use $\lfloor x \rfloor$ to represent the floor of x , ie the largest integer less than or equal to x , for example $\lfloor -0.5 \rfloor = -1$.

We also make extensive use of the following “almost standard” notation, which we define here precisely in order to eliminate any ambiguity over edge cases:

- The fractional part function $\{x\}$ maps x to $x - \lfloor x \rfloor \in [0, 1)$, for example $\{-1.25\} = 0.75$
- $f(x) = O(g(x))$ as $x \rightarrow C \in [-\infty, +\infty]$ means that there is a positive real constant M and a neighbourhood $N(C)$ such that $|f(x)| < M|g(x)|$ for $x \in N(C)$, for example $1/(c-x) = O(1/x)$ as $x \rightarrow +\infty$. Normally C is 0 or $+\infty$, and will be omitted if clear from the context.

4.1.1. Generalised notation for sums and products. As usual we will define the empty sum to have the value 0, and the empty product to have the value 1.

Given a summable sequence (a_r) we will find it useful to define a generalised summation notation $\sum_{r=x}^y a_r$ to include real (rather than integer) upper and lower bounds x, y . We do this by defining the step function $f(t) = a_r$ for $t \in [r, r+1)$, and then $\sum_{r=x}^y a_r = \int_x^y f(t) dt$. If $f(t) > 0$ on $[x, y)$ we define the multiplicative analogue as $\prod_{r=x}^y a_r = \exp \int_x^y \log f(t) dt$. For example for odd integers $n = 2k+1$:

$$\sum_1^{n/2} a_r = \sum_1^k a_r + \frac{1}{2} a_{k+1}$$

Note that for integer x, y the definitions coincide with normal summation and product notation.

4.2. Special sequences used in this paper. In addition to the Fibonacci sequence $(F_i) = (0, 1, 1, 2, 3, 5, 8, \dots)$, we make extensive use of a number of derived sequences which we define here for convenience. Note we only define them for integer n, t and $n \geq 1$.

$$(4.1) \quad s_{nt} = 2 \sin \pi \left(\frac{t}{F_n} - \omega^n \left(\frac{[tF_{n-1}]}{F_n} - 1/2 \right) \right)$$

$$(4.2) \quad \xi_{nt} = \begin{cases} \frac{[tF_{n-1}]}{F_n} - \frac{1}{2} & ([t] \neq 0 \bmod F_n) \\ 0 & ([t] = 0 \bmod F_n) \end{cases}$$

$$(4.3) \quad \xi_{\infty t} = \{t\omega\} - \frac{1}{2}$$

$$(4.4) \quad h_{nt} = \cot \frac{\pi t}{F_n} \sin \pi \omega^n \xi_{nt} \quad ([t] \neq 0 \bmod F_n)$$

Note that $s_{nt} = 2 \sin \pi (t/F_n - \omega^n \xi_{nt})$ when $[t] \neq 0 \bmod F_n$, but not when $[t] = 0 \bmod F_n$ due to the alternative definition of ξ_{nt} . This reflects the fact that the two sequences play very different roles, and each definition makes sense in its own context. We have also chosen to leave h_{nt} undefined for $[t] = 0 \bmod F_n$.

Lemma 4.1. *For the sequences $s_{nt}, \xi_{nt}, \xi_{\infty t}$ defined above:*

- (1) *For fixed $n \geq 1$, the sequences $|s_{nt}|, \xi_{nt}, h_{nt}$ are periodic sequences of period F_n , and further s_{nt}, ξ_{nt} are both odd sequences in t (ie of the form $a_t = -a_{-t}$) and h_{nt} is an even sequence in t (ie of the form $a_t = a_{-t}$).*
- (2) *Both $|\xi_{nt}| < 1/2$ and $|\xi_{\infty t}| < 1/2$ with the exception of $\xi_{\omega 0} = -1/2$.*
- (3) *In the range $0 \leq t \leq F_n - 1$, $s_{nt} \geq s_{n0} > 0$ with equality only at $t = 0$. For any t , $s_{n, F_n+t} = -s_{nt}$*
- (4) *$\xi_{n, F_n-t} = -\xi_{nt}$ whereas $s_{n, F_n-t} = s_{nt}$, and $h_{n, F_n-t} = h_{nt}$*
- (5) *For $1 \leq t \leq F_{n-1}$ we have $\xi_{nt} = \xi_{\infty t} + O(\omega^n)$ and $\lim_{n \rightarrow \infty} \xi_{nt} = \xi_{\infty t}$*

Proof.

□

- (1) Note that $\{tF_{n-1}/F_n\}$ is of period F_n , and the periodicity results follow, noting also that $|\sin \pi x|$ is of period 1. Also we have $\{-x\} = 1 - \{x\}$, from which the oddness of ξ_{nt} immediately follows. The oddness of s_{nt} then follows from the oddness of $\sin x$. The evenness of h_{nt} follows from the oddness of both \cot and \sin .
- (2) Both results follow from $0 < \{x\} < 1$ unless $x = 0$. But $\{tF_{n-1}/F_n\} = 0$ only for $[t] = 0 \bmod F_n$ and then $\xi_{nt} = 0$. And $\{t\omega\} = 0$ only for $t = 0$.
- (3) For $t = 0$ we have $s_{nt} = s_{n0} = 2 \sin \pi \omega^n / 2 > 0$. For $n = 1, 2$ the only possibility is $t = 0$, but for $n \geq 3$ and $1 \leq t \leq F_n - 1$, then $s_{nt} \geq s_{n1} = 2 \sin \pi (F_n^{-1} - \omega^n \xi_{nt})$. But $|\xi_{nt}| < 1/2$, and $F_n^{-1} = \sqrt{5}\omega^n / (1 - (-1)^n \omega^{2n}) > 2\omega^n$ so that $s_{n1} > s_{n0} > 0$. The second part follows by noting that substituting $F_n + t$ in s_{nt} simply adds π to the argument of the sine function.
- (4) These now follow easily from the previous results
- (5) Since $t \neq 0$, we have $\xi_{nt} - \xi_{wt} = \{tF_{n-1}/F_n\} - \{t\omega\}$. Now by (4.8) $t\omega = tF_{n-1}/F_n - t(-\omega^n)/F_n$, but $t < F_n$ so $|t\omega - tF_{n-1}/F_n| < \omega^n < 1/F_n$ which means $\{t\omega\}$ is always inside the interval $\{tF_{n-1}/F_n\} \pm 1/F_n$, and we can deduce that $|\xi_{nt} - \xi_{wt}| < \omega^n$. The results follow.

4.3. Inequalities. We gather here various inequalities which we will need during the main proofs.

Lemma 4.2. *For x in $(0, \pi/2)$ we have $2x/\pi < \sin x < x$*

Proof. The derivative of $f(x) = x - \sin x$ is $1 - \cos x$ which is positive. So $f(x)$ is increasing, and $f(0) = 0$, and the right side inequality follows. For the left side we use the fact that $\sin x$ is convex in this interval and hence lies above the line segment joining $(0, 0)$ and $(\pi/2, 1)$. But this is $2x/\pi$. □

Lemma 4.3. For $n \geq 2$, let (a_t) $t = 1 \dots n$ be a sequence of real numbers satisfying $|a_t| < 1$ with $A = \sum |a_t| < 1$. Then

$$1 - A < \prod_{t=1}^n (1 + a_n) < \frac{1}{1 - A}$$

Proof. $\prod_{t=1}^n (1 + a_t) \geq \prod_{t=1}^n (1 - |a_t|)$. Then $\prod_{t=1}^n (1 - |a_t|) > 1 - A$ is clearly true for $n = 2$ and the left hand side of the result follows by induction.

Similarly $\prod_{t=1}^n (1 + a_t) \leq \prod_{t=1}^n (1 + |a_t|)$. Then $\prod_{t=1}^n (1 + |a_t|) < \sum_{r=0}^n (\sum |a_t|)^r = (1 - A^{n+1})/(1 - A)$ proving the right hand side of the result. \square

4.4. Results on Fibonacci numbers. We will make use of some standard results about Fibonacci numbers $F_n = (0, 1, 1, 2, 3, 5, 8 \dots)$ defined for $n \geq 0$ by $F_0 = 0, F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$

$$(4.5) \quad F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

$$(4.6) \quad F_n \text{ is even iff } n = 3k \text{ for some } k \geq 0$$

$$(4.7) \quad F_n = \frac{1}{\sqrt{5}} (\omega^{-n} - (-\omega)^n) = \frac{\omega^{-n}}{\sqrt{5}} + O(F_n^{-1})$$

$$(4.8) \quad F_n \omega = F_{n-1} - (-\omega)^n \Leftrightarrow \omega = \frac{F_{n-1}}{F_n} - \frac{(-\omega)^n}{F_n} = \frac{F_{n-1}}{F_n} + O(\omega^{2n})$$

We give here some other simple results we will need later.

Lemma 4.4. For $n \geq 1$, defining $[0]^{-1} = [0] \bmod 1$, the inverse of $F_{n-1} \bmod F_n$ exists and is $[(-1)^n F_{n-1}]$

Proof. From (??) we have $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ and substituting the definition $F_{n+1} = F_n + F_{n-1}$ gives $F_{n-1}^2 \equiv (-1)^n \bmod F_n$, whence $F_{n-1} \cdot (-1)^n F_{n-1} \equiv 1 \bmod F_n$. \square

4.4.1. Representation by Fibonacci numbers. We will use a representation of $k \geq 1$ as a sum of Fibonacci numbers via the following definition:⁷

Definition 4.5. Fibonacci summation algorithm

For $n \geq 0$, define the *Fibonacci floor* $F(n)$ to be the largest Fibonacci number $F_i \leq n$, for example $F(0) = F_0, F(1) = F_2, F(7) = F_5$.

We now define the *Fibonacci sum* $S(n)$ to be the series defined recursively by $S(n) = F(n) + S(n - F(n))$, $S(0) = F_0$. For example $S(7) = F_5 + F_3 + F_0 = 5 + 2 + 0$.

We define the Fibonacci length $F_L(n)$ of n to be the length of the Fibonacci sum ignoring the terminal F_0 element. For example the Fibonacci lengths of 0, 7 are 0, 2 respectively.

Since the algorithm is deterministic, it provides a unique representation for each n . Further, since for $i \geq 2$ we have $F_i = F(n) \leq n < F_{i+1}$, and since $F_{i+1} = F_i + F_{i-1}$, we also have $n - F(n) < F_{i+1} - F_i = F_{i-1}$, so that if F_i is in the representation, F_{i-1} is not. Finally note that since $F_2 = F_1 = 1$, we never have $F(n) = F_1$.

Definition 4.6. The binary Fibonacci representation $n = \sum_{s=1}^m b_s F_s$ for $n \geq 1$

We will find it useful to translate the sum $S(n)$ into the equivalent representation $\sum_{s=1}^m b_s F_s$ where $b_s = 1$ if F_s is in $S(n)$, and $b_s = 0$ otherwise, and $F_m = F(n)$.

Note that the Fibonacci length $F_L(n)$ of n is now $\sum_{s=1}^m b_s \leq m - 1$ since F_1 is never in $S(n)$, but we can improve this result as follows:

⁷We give here our own recursive definition which we prefer to classical definitions as it obviates the need to prove existence or uniqueness of the resulting representation. It also provides for straightforward translation into modern programming languages, and is easily extended to an Ostrowski representation. However the result is of course equivalent to the Zeckendorf representation.

Lemma 4.7. *If $n \geq 1$ has the representation $\sum_{s=1}^m b_s F_s$ then:*

$$(4.9) \quad m \leq \lfloor (\log n + 1) / \log(1 + \omega) \rfloor$$

$$(4.10) \quad F_L(n) \leq \lfloor (\log n + 1) / \log(2 + \omega) \rfloor$$

Proof. Since $F_m = F(n) \leq n$, we use from (4.7) $F_m = (\omega^{-m} - (-\omega)^m) / \sqrt{5}$ and we deduce (using $\log(1+x) < x$, and $\omega^{-1} = 1 + \omega$):

$$(4.11) \quad \begin{aligned} m &= \max\{j : \omega^{-j} \leq \sqrt{5}n + (-\omega)^j\} \\ &= \max\{j : j \leq \log(\sqrt{5}n + (-\omega)^j) / \log(1 + \omega)\} \\ &\leq \lfloor (\log n + 1) / \log(1 + \omega) \rfloor \end{aligned}$$

Now the Fibonacci length $F_L(n) = \sum_{s=1}^m b_s \leq m - 1$, but in fact we can do better than this. Since $S(n)$ contains no two consecutive F_i we have:

$$(4.12) \quad F_L(n) \leq \lfloor m/2 \rfloor$$

The result follows using $(1 + \omega)^2 = 2 + \omega$. □

5. THE DECOMPOSITION $Q_n = A_n B_n C_n$

As described in section 3.1, we develop a decomposition of Q_n into a product of three other products, each of which converges to a positive constant. We shall prove the convergence of the first of these products within this section (as it is very straightforward), and the other two we shall deal with in subsequent sections.

Our central motivation here is to substitute the Fibonacci identity $\omega = (F_{n-1}/F_n) - (-\omega)^n/F_n$ (see (4.8)) into the definition of Q_n and hence express $|\prod 2 \sin \pi r \omega|$ as a perturbation of the rational sine product $|\prod 2 \sin \pi r (F_{n-1}/F_n)|$, the latter product being equal to F_n (see (??)). This reduces the problem to one of demonstrating that the perturbation function itself has suitable behaviour, and this proves equivalent to showing that the product $B_n C_n$ converges as n grows. However rather than treating $B_n C_n$ as a single product, it is simpler to prove separately that each of B_n and C_n converge.

The substitution above gives us $Q_n = |\prod 2 \sin \pi r ((F_{n-1}/F_n) - (-\omega)^n/F_n)|$ which is a perturbation of the the argument in each term of $|\prod 2 \sin \pi r (F_{n-1}/F_n)|$ by a delta of $-r(-\omega)^n/F_n$. The sum of these deltas is non-zero, but some of the techniques we shall use to prove the convergence of B_n, C_n require that the sum of the deltas is 0. Fortunately, as we shall see, we can fix this by re-basing the arguments to result in a delta of $\omega^n(r/F_n - 1/2)$ - which then provides a zero sum for the deltas. This is most economically achieved once and for all at the beginning of our proof, and will simplify later proofs at the cost introducing a non-intuitive first step below. However once done, we proceed to make the substitution for ω , and the decomposition then follows naturally.

Lemma 5.1. *For $n \geq 1$ and $s_{nt} = 2 \sin \pi \left(t/F_n - \omega^n \left(\frac{F_{n-1}t}{F_n} - \frac{1}{2} \right) \right)$ we have $Q_n = \left| \prod_{r=1}^{F_n} (2 \sin \pi r \omega) \right| = A_n B_n C_n$ where:*

$$(5.1) \quad A_n = 2F_n \sin \pi \omega^n \rightarrow \frac{2\pi}{\sqrt{5}}$$

$$(5.2) \quad B_n = \left(\prod_{t=1}^{F_n-1} \frac{s_{nt}}{2 \sin \pi \frac{t}{F_n}} \right)$$

$$(5.3) \quad C_n = \prod_{t=1}^{F_n/2} \left(1 - \frac{s_{n0}^2}{s_{nt}^2} \right)$$

We first deal with the convergence of A_n by observing that since $\omega < 1$, we have $A_n = 2F_n \sin \pi \omega^n \sim 2F_n \pi \omega^n$ and the result follows by (4.8).

We start the main proof by carrying out the step discussed above to re-base our perturbation deltas. First we exploit the symmetry of the sine function around $\pi/2$, observing that a change of variables $r \mapsto F_n - r$ gives us $\prod_{r=1}^{F_n-1} (2 \sin \pi r \omega) = \prod_{r=1}^{F_n-1} (2 \sin \pi (F_n - r) \omega)$ and hence for any $n \geq 1$, using the product of sines formula:

$$\begin{aligned}
 Q_n^2 &= (2 \sin \pi F_n \omega)^2 \prod_{r=1}^{F_n-1} (2 \sin \pi r \omega) (2 \sin \pi (F_n - r) \omega) \\
 (5.4) \quad &= (2 \sin \pi F_n \omega)^2 \prod_{r=1}^{F_n-1} 2 (\cos \pi (F_n - 2r) \omega - \cos \pi F_n \omega)
 \end{aligned}$$

We can now use identity (4.8) and the cosine double angle formula to obtain:

$$\begin{aligned}
 Q_n^2 &= (2 \sin \pi \omega^n)^2 \prod_{r=1}^{F_n-1} 2(-1)^{F_n-1} (\cos \pi ((-\omega)^n + 2r\omega) - \cos \pi (-\omega)^n) \\
 &= (2 \sin \pi \omega^n)^2 (-1)^{(F_n-1)F_n-1} \prod_{r=1}^{F_n-1} 4 \left(\sin^2 \frac{1}{2} \pi \omega^n - \sin^2 \pi (r\omega + \frac{1}{2}(-\omega)^n) \right) \\
 (5.5) \quad &= (2 \sin \pi \omega^n)^2 (-1)^{(F_n-1)(F_n-1+1)} \prod_{r=1}^{F_n-1} 4 \left(\sin^2 \pi (r\omega + \frac{1}{2}(-\omega)^n) - \sin^2 \frac{1}{2} \pi \omega^n \right)
 \end{aligned}$$

Now if F_n is odd then $F_n - 1$ is even, and if F_n is even then by (4.6) $F_{n-1} + 1$ is even, and so for any n we have $(-1)^{(F_n-1)(F_n-1+1)} = 1$. We have therefore shown that

$$(5.6) \quad Q_n^2 = (2 \sin \pi \omega^n)^2 \prod_{r=1}^{F_n-1} 4 \left(\sin^2 \pi (r\omega + \frac{1}{2}(-\omega)^n) - \sin^2 \frac{1}{2} \pi \omega^n \right)$$

This completes the re-basing step. We are now ready to develop the expression for Q_n as a perturbation of the rational sine product $|\prod 2 \sin \pi r (F_{n-1}/F_n)|$.

The product above is empty for $n = 1, 2$. For $n \geq 3$ we develop the second sine term above by substituting the Fibonacci identity and then using (4.8) to obtain

$$(5.7) \quad \sin \pi (r\omega + (-\omega)^n/2) = \sin \pi \left(\frac{rF_{n-1}}{F_n} - (-\omega)^n \left(\frac{r}{F_n} - \frac{1}{2} \right) \right)$$

Substituting the residue $t = [rF_{n-1}]$ in the right hand term and using Lemma 4.4 we obtain

$$(5.8) \quad \sin \pi (r\omega + (-\omega)^n/2) = \pm \sin \pi \left(\frac{t}{F_n} - (-\omega)^n \left(\frac{[(-1)^n F_{n-1} t]}{F_n} - \frac{1}{2} \right) \right)$$

Now observe that $x \mapsto \{x\} - 1/2$ is an odd function (for non-integer x), and we use this fact to simplify the right side to obtain finally for every $1 \leq r \leq F_n - 1$

$$(5.9) \quad \left| \sin \pi (r\omega + (-\omega)^n/2) \right| = \left| \sin \pi \left(\frac{t}{F_n} - \omega^n \left(\frac{[F_{n-1} t]}{F_n} - \frac{1}{2} \right) \right) \right|$$

Now the right hand side is $|s_{nt}|$, and for $1 \leq r \leq F_n - 1$ we also have $1 \leq t \leq F_{n-1}$. In this range for t we have $s_{nt} > 0$ by Lemma 4.1. This gives us for $1 \leq s, t \leq F_n - 1$:

$$(5.10) \quad \left| \sin \pi (r\omega + (-\omega)^n/2) \right| = s_{nt}$$

If we further observe that for $1 \leq r \leq F_n - 1$, $t = [rF_{n-1}]$ runs through a complete set of non-zero residues, so we can rewrite (5.6) for $n \geq 1$ as:

$$\begin{aligned}
 Q_n^2 &= (2 \sin \pi \omega^n)^2 \prod_{t=1}^{F_n-1} (s_{nt}^2 - s_{n0}^2) \\
 (5.11) \quad &= (2 \sin \pi \omega^n)^2 \left(\prod_{t=1}^{F_n-1} s_{nt} \right)^2 \prod_{t=1}^{F_n-1} \left(1 - \frac{s_{n0}^2}{s_{nt}^2} \right)
 \end{aligned}$$

We have almost proved Lemma 5.1. To obtain the final result, we use the standard result that for any p relatively prime to $q \geq 1$

$$(5.12) \quad \prod_{r=1}^{q-1} 2 \sin\left(\frac{\pi r p}{q}\right) = q$$

(For a particularly elegant proof see Knill (2012) [8]). From this result we obtain $\prod_{t=1}^{F_n-1} 2 \sin \pi \frac{t}{F_n} = F_n$ and the result follows (using $s_{nt} = s_{n(F_n-t)}$ from Lemma 4.1).

6. THE CONVERGENCE OF $C_n = \prod_{t=1}^{F_n/2} \left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right)$

In this step we show C_n converges to a strictly positive constant. This is not as straightforward as it appears at first sight as there are terms in s_{nt} which oscillate about 0 but which are not alternating. We therefore cannot assume that C_n is decreasing. Fortunately we are able to compare C_n with a closely related sequence which *is* decreasing and therefore converges.

Theorem 6.1. *The sequence $C_n = \prod_{t=1}^{F_n/2} \left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right)$ converges to*

$$\prod_{t=1}^{\infty} \left(1 - \frac{1}{20 \left(t - \frac{1}{\sqrt{5}} (\{t\omega\} - \frac{1}{2}) \right)^2} \right) \simeq 0.928$$

For $n \in \{0, 1, 2\}$ the product defining C_n is empty and $C_n = 1$. For the rest of this section we will assume $n \geq 3$, and so by Lemma 4.1 we have for $1 \leq t \leq F_n - 1$ that $s_{nt} > s_{n0} > 0$

Hence we have $0 < (1 - s_{n0}^2/s_{nt}^2) < 1$ for every term in C_n , and so $0 < C_n < 1$ for $n \geq 3$.

At this point we need to establish some estimates for the terms s_{n0}/s_{nt} . First we develop some general estimates valid for all $0 \leq t < F_n$.

For $t = 0$ we have

$$(6.1) \quad s_{n0} = 2 \sin \pi(\omega^n/2) = \pi \omega^n (1 + O(\omega^{2n}))$$

For $1 \leq t \leq F_n/2$, from (4.8) $F_n^{-1} = \sqrt{5} \omega^n (1 + O(\omega^{2n}))$, and from (4.8) $F_{n-1}/F_n = \omega + O(\omega^{2n})$ and so:

$$\begin{aligned}
 s_{nt} &= 2 \sin \pi \left(\left(t \sqrt{5} \omega^n (1 + O(\omega^{2n})) \right) - \omega^n \left(\{t\omega\} + tO(\omega^{2n}) - \frac{1}{2} \right) \right) \\
 (6.2) \quad &= 2 \sin \pi \omega^n t \left(\sqrt{5} - \frac{1}{t} \left(\{t\omega\} - \frac{1}{2} \right) + O(\omega^{2n}) \right)
 \end{aligned}$$

Now let $q = \lceil \omega^{-3n/5} \rceil^8$. For $t \geq q$ we use $(\pi/2) \sin x > x$ (from Lemma 4.2) in (6.2) to give us for large enough n :

$$\begin{aligned}
 \frac{s_{n0}}{s_{nt}} &< \frac{\pi \omega^n (1 + O(\omega^{2n}))}{(2/\pi) 2\pi \omega^n t \left(\sqrt{5} - \frac{1}{t} (\{t\omega\} - \frac{1}{2}) + O(\omega^{2n}) \right)} \\
 &< \frac{\pi (1 + O(\omega^{2n}))}{4q \left(\sqrt{5} - q^{-1} (\{t\omega\} - \frac{1}{2}) + O(\omega^{2n}) \right)} \\
 &< \frac{\pi (1 + O(q^{-1}))}{4\sqrt{5}q} \\
 (6.3) \qquad &= O(q^{-1})
 \end{aligned}$$

Now choose $q \leq q_1 < q_2 \leq F_n/2$. We can now use from Lemma 4.3 $\prod(1 - a_n) > 1 - \sum |a_n|$ to obtain:

$$\begin{aligned}
 1 &> \prod_{t=q_1}^{q_2} \left(1 - \frac{s_{n0}^2}{s_{nt}^2} \right) > \prod_{t=q_1}^{F_n/2} \left(1 - \frac{s_{n0}^2}{s_{nt}^2} \right) \\
 &> 1 - \sum_{t=q_1}^{F_n/2} O(q^{-2}) > 1 - F_n O(\omega^{6n/5}) \\
 (6.4) \qquad &= 1 - O(\omega^{n/5})
 \end{aligned}$$

Now we consider the case of $t < q$. From (6.2) we have $s_{nt} = 2 \sin \pi \omega^n t \left(\sqrt{5} - (\{t\omega\} - \frac{1}{2})/t + O(\omega^{2n}) \right)$, and the largest term in the argument of the sine function is then $O(\omega^n q) = O(\omega^{2n/5})$, so that for large enough n we can make the argument as small as we like. So we can use $\sin x = x + O(x^3)$ to give us:

$$\begin{aligned}
 s_{nt} &= 2\pi \omega^n t \left(\sqrt{5} - \frac{1}{t} \left(\{t\omega\} - \frac{1}{2} \right) + O(\omega^{2n}) \right) + O(\omega^{6n/5}) \\
 (6.5) \qquad &= 2\sqrt{5}\pi \omega^n \left(t - \frac{1}{\sqrt{5}} \left(\{t\omega\} - \frac{1}{2} \right) + O(\omega^{n/5}) \right)
 \end{aligned}$$

We put $u_t = 2\sqrt{5} \left(t - \frac{1}{\sqrt{5}} (\{t\omega\} - \frac{1}{2}) \right)$. Using (6.1) we get:

$$(6.6) \qquad \frac{s_{n0}}{s_{nt}} = \frac{(1 + O(\omega^{n/5}))}{u_t}$$

Hence we can write:

$$\begin{aligned}
 \prod_{t=1}^q \left(1 - \frac{s_{n0}^2}{s_{nt}^2} \right) &= \prod_{t=1}^q \left(1 - \frac{1}{u_t^2} - \frac{O(\omega^{n/5})}{u_t^2} \right) \\
 (6.7) \qquad &= \prod_{t=1}^q \left(1 - \frac{1}{u_t^2} \right) \prod_{t=1}^q \left(1 - \frac{O(\omega^{n/5})}{u_t^2 - 1} \right)
 \end{aligned}$$

Now $\sum 1/(u_t^2 - 1)$ converges (by comparison with $\sum 1/t^2 = \pi^2/6$) and so $\sum \frac{O(\omega^{n/5})}{u_t^2 - 1} = O(\omega^{n/5})$, so by Lemma 4.3:

$$(6.8) \qquad \prod_{t=1}^q \left(1 - \frac{O(\omega^{n/5})}{u_t^2 - 1} \right) = 1 + O(\omega^{n/5})$$

Similarly $\sum 1/u_t^2$ also converges, but for this series we need more information about the limit which we obtain as follows:

⁸Here $3/5$ is chosen to optimise convergence, though other values are possible.

$$\begin{aligned}
\sum_{t=1}^{\infty} \frac{1}{u_t^2} &< \frac{1}{u_1^2} + \sum_{t=2}^{\infty} \frac{1}{20(t-1)^2} \\
&< 0.056 + \pi^2/120 \\
(6.9) \quad &< 0.138
\end{aligned}$$

We now put $U_q = \prod_{t=1}^q \left(1 - \frac{1}{u_t^2}\right) > 1 - \sum_{t=1}^q 1/u_t^2 > 0.862$. Note that U_q is a descending sequence and bounded below, and so converges to some constant $U_{\infty} > 0.862$. (In fact we compute $U_{\infty} \simeq 0.928$). And

$$\begin{aligned}
1 > \frac{U_{\infty}}{U_q} &= \prod_{t=q+1}^{\infty} \left(1 - \frac{1}{u_t^2}\right) > 1 - \frac{1}{20} \sum_{t=q+1}^{\infty} \frac{1}{(t-1)^2} \\
&= 1 - O(q^{-1})
\end{aligned}$$

Finally:

$$\begin{aligned}
C_n &= \prod_{t=1}^{F_n/2} \left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right) = \prod_{t=1}^q \left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right) \prod_{t=q+1}^{F_n/2} \left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right) \\
&= U_{\infty} (1 + O(q^{-1})) (1 + O(\omega^{n/5})) (1 - O(\omega^{n/5})) \\
&= U_{\infty} (1 + O(\omega^{n/5}))
\end{aligned}$$

Hence

$$(6.10) \quad \lim_{n \rightarrow \infty} C_n = U_{\infty} = \prod_{t=1}^{\infty} \left(1 - \frac{1}{20 \left(t - \frac{1}{\sqrt{5}} (\{t\omega\} - \frac{1}{2})\right)^2}\right)^2 \simeq 0.928$$

7. THE CONVERGENCE OF $B_n = \prod_{t=1}^{F_n-1} s_{nt} / (2 \sin \pi t / F_n)$

In this step we show that B_n converges to a strictly positive limit. In the last section we saw that the proof of convergence was complicated by the presence of non-alternating oscillations in sign. We were able to circumvent this problem by relating the product to one which converged absolutely, and involved a product of square terms $\prod(1 - 1/r^2)$. In this section we are unable to do this as the absolute product behaves like $\prod(1 + 1/r)$ and diverges. The convergence is therefore conditional and we are forced to estimate the compound effects of the signed differences.

Theorem 7.1. *The sequence $\log B_n$ converges to a finite limit, and the sequence B_n to a strictly positive limit*

We start by examining each term for $1 \leq t \leq F_n - 1$:

$$\begin{aligned}
\frac{s_{nt}}{2 \sin \pi t / F_n} &= \frac{2 \sin \pi(t/F_n - \omega^n \xi_{nt})}{2 \sin \pi t / F_n} \\
&= \cos \pi \omega^n \xi_{nt} - \cot \frac{\pi t}{F_n} \sin \pi \omega^n \xi_{nt} \\
(7.1) \quad &= 1 - 2 \sin^2 \frac{\pi}{2} \omega^n \xi_{nt} - \cot \frac{\pi t}{F_n} \sin \pi \omega^n \xi_{nt}
\end{aligned}$$

Put $\alpha_{nt} = 2 \sin^2 \frac{1}{2} \pi (\omega^n \xi_{nt})$ and $h_{nt} = \cot \pi \left(\frac{t}{F_n}\right) \sin \pi (\omega^n \xi_{nt})$ so that $B_n = \prod_{t=1}^{F_n-1} (1 - \alpha_{nt} - h_{nt})$.

We first need an estimate for h_{nt} . Using $\cot x < 1/x$ in $(0, \pi/2)$ and $|\xi_{nt}| < 1/2$ gives us:

$$\begin{aligned}
|h_{nt}| &= \cot\left(\frac{\pi t}{F_n}\right) |\sin \pi(\omega^n \xi_{nt})| \\
&< \frac{F_n}{\pi t} \cdot \pi \omega^n |\xi_{nt}| \\
&< \frac{1}{2\sqrt{5}t} (1 - (-1)^n \omega^{2n}) \\
(7.2) \quad &< \frac{1}{4t} \text{ for any } 1 \leq t \leq F_n/2
\end{aligned}$$

Also since $|\xi_{nt}| < 1/2$ we have $0 < \alpha_{nt} < \pi^2 \omega^{2n}/8$. Consequently $\log(1 - \alpha_{nt} - h_{nt}) = \log(1 - h_{nt}) + O(\omega^{2n})$ and we can sum over t to obtain:

$$(7.3) \quad \left| \log B_n - \sum_{t=1}^{F_n-1} \log(1 - h_{nt}) \right| = O(\omega^n)$$

Writing $B_n^* = \prod_{t=1}^{F_n-1} (1 - h_{nt})$, this gives us:

$$(7.4) \quad B_n \sim B_n^*$$

We proceed to investigate the product B_n^* . We start by observing:

$$(7.5) \quad \log B_n^* = \sum_{t=1}^{F_n-1} \log(1 - h_{nt}) = - \sum_{t=1}^{F_n-1} \sum_{k=1}^{\infty} \frac{1}{k} h_{nt}^k$$

Using (from Lemma 4.1) the symmetry $h_{nt} = h_{n(F_n-t)}$, we obtain:

$$(7.6) \quad \log B_n^* = -2 \sum_{t=1}^{F_n/2} \sum_{k=1}^{\infty} \frac{1}{k} h_{nt}^k = -2 \left(\sum_{t=1}^{F_n/2} h_{nt} + \sum_{t=1}^{F_n/2} \sum_{k=2}^{\infty} \frac{1}{k} h_{nt}^k \right)$$

7.1. Convergence of $\sum_{t=1}^{F_n/2} \sum_{k=2}^{\infty} \frac{1}{k} h_{nt}^k$. Examining the right hand sum we find we can immediately take limits, using (7.2):

$$\lim_{n \rightarrow \infty} \sum_{t=1}^{F_n/2} \sum_{k=2}^{\infty} \frac{1}{k} |h_{nt}^k| = \sum_{t=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{k} |h_{nt}^k| < \sum_{t=1}^{\infty} \frac{h_{nt}^2}{1 - |h_{nt}|} < \frac{\pi^2}{72}$$

Hence the sum above is absolutely convergent, and hence convergent to a limit we denote L_2^B , ie:

$$(7.7) \quad \lim_{n \rightarrow \infty} \sum_{t=1}^{F_n/2} \sum_{k=2}^{\infty} \frac{1}{k} h_{nt}^k = L_2^B$$

7.2. Convergence of $\sum_{t=1}^{F_n/2} h_{nt}$. We are left in (7.6) with estimating the first sum $\sum_{t=1}^{F_n/2} h_{nt}$. Our estimate of $|h_{nt}| < 1/4t$ is not good enough to help us here as its sum is the divergent harmonic series.

Put $H_n = \sum_{t=1}^{F_n/2} h_{nt} = \sum_{t=1}^{F_n/2} \cot\left(\frac{\pi t}{F_n}\right) \sin \pi(\omega^n \xi_{nt})$ and $H_n^* = \sum_{t=1}^{F_n/2} \cot\left(\frac{\pi t}{F_n}\right) \sin \pi(\omega^n \xi_{\infty t})$ (where in H_n^* we have simply replaced ξ_{nt} with $\xi_{\infty t}$). Note that for $1 \leq t \leq F_n - 1$ we have $\xi_{nt} - \xi_{\infty t} = t(-\omega)^n/F_n$ so that

$$(7.8) \quad H_n - H_n^* = \sum_{t=1}^{F_n/2} \cot\left(\frac{\pi t}{F_n}\right) \pi \omega^n \frac{t(-\omega)^n}{F_n} (1 + O(\omega^{2n}))$$

Note that for $x \in (0, \pi/2]$ we have $\cot x < x^{-1}$ and so

$$\begin{aligned}
 |H_n - H_n^*| &< \sum_{t=1}^{F_n/2} \left(\frac{\pi t}{F_n} \right)^{-1} \pi \omega^n \frac{t(\omega)^n}{F_n} (1 + O(\omega^{2n})) = \omega^{2n} \sum_{t=1}^{F_n/2} (1 + O(\omega^{2n})) \\
 (7.9) \qquad &= \frac{\omega^n}{\sqrt{5}} + O(\omega^{3n})
 \end{aligned}$$

so that $H_n - H_n^* \rightarrow 0$. We now focus on H_n^* . For the next step we will need to revert to summation using integer limits. To do this note that if F_n is even then $h_{F_n/2} = \cot \pi/2 \sin \pi (\omega^n \xi_{F_n/2}) = 0$ so we can ignore this term. So now we can put $M_n = \lfloor (F_n - 1)/2 \rfloor$ and use summation by parts to obtain:

$$\begin{aligned}
 H_n^* &= \sum_{t=1}^{M_n} \cot \left(\frac{\pi t}{F_n} \right) \sin \pi (\omega^n \xi_{\omega t}) \\
 (7.10) \qquad &= \sum_{t=1}^{M_n-1} \left(\cot \left(\frac{\pi t}{F_n} \right) - \cot \left(\frac{\pi(t+1)}{F_n} \right) \right) \sum_{s=1}^t \sin \pi (\omega^n \xi_{\omega s}) + \cot \left(\frac{\pi M_n}{F_n} \right) \sin \pi (\omega^n \xi_{\omega M_n})
 \end{aligned}$$

Recalling $|\xi_{\omega t}| < 1/2$, the trailing term is easily estimated as:

$$(7.11) \qquad \left| \cot \left(\frac{\pi M_n}{F_n} \right) \sin \pi (\omega^n \xi_{\omega M_n}) \right| < \left(\frac{\pi}{2F_n} \right) \left(\frac{\pi \omega^n}{2} \right) + O(\omega^{4n}) = O(\omega^{2n})$$

We can now take limits on (7.10) to obtain, writing $\alpha = \pi/F_n$, $C_{nt} = \cot t\alpha - \cot(t+1)\alpha$ and $S_{nt} = \sum_{s=1}^t \sin \pi (\omega^n \xi_{\omega s})$:

$$(7.12) \qquad \lim_{n \rightarrow \infty} H_n^* = \lim_{n \rightarrow \infty} \sum_{t=1}^{M_n-1} C_{nt} S_{nt}$$

7.2.1. *The order of the cotangent difference.* We estimate the cotangent difference as follows:

$$\begin{aligned}
 0 < C_{nt} = \cot t\alpha - \cot(t+1)\alpha &= \frac{\sin(t+1)\alpha \cos t\alpha - \cos(t+1)\alpha \sin t\alpha}{\sin t\alpha \sin(t+1)\alpha} \\
 &= \frac{2 \sin \alpha}{\cos \alpha - \cos(2t+1)\alpha} \\
 (7.13) \qquad &= \frac{\sin \alpha}{\sin^2(t + \frac{1}{2})\alpha - \sin^2 \alpha}
 \end{aligned}$$

Expanding α , and noting from Lemma 4.2 $(\pi/2) \sin x > x$ for $x \in (0, \pi/2)$ we get:

$$\begin{aligned}
 0 < C_{nt} &< \frac{\pi F_n (1 + O(F_n^{-2}))}{(2t+1)^2 - 1} \\
 (7.14) \qquad &< \frac{\pi F_n (1 + O(F_n^{-2}))}{2t^2}
 \end{aligned}$$

7.2.2. *The order of the partial sums* $S_{nt} = \sum_{s=1}^t \sin \pi (\omega^n \xi_{\omega s})$. As can be seen from the figure, the partial sums S_{nt} appear to grow slowly. In this step we bound this growth by establishing an estimate for S_{nt} in terms of t and n . However we will also need introduce a generalised $S_{nt}(\theta)$ in order to accommodate a dependency on a starting phase angle θ . Our basic approach will be to find an estimate for $S_{nF_k}(\theta)$ and then express $S_{nt}(0) = S_{nt}$ as a sum of terms involving $S_{nF_k}(\theta)$.

Recall from 4.5 that we can represent $t \geq 1$ as a Fibonacci sum $t = \sum_{s=1}^m b_s F_s$ where $m(t)$ is the largest integer such that $F_m \leq t$. (This is an approach which has been used by several researchers, eg Knill [8]). Define $t_m = 0$, and for $0 \leq s \leq m-1$ define $t_s = t_{s+1} + b_{s+1} F_{s+1} = \sum_{u=s+1}^m b_u F_u$ so that $t_0 = t$.

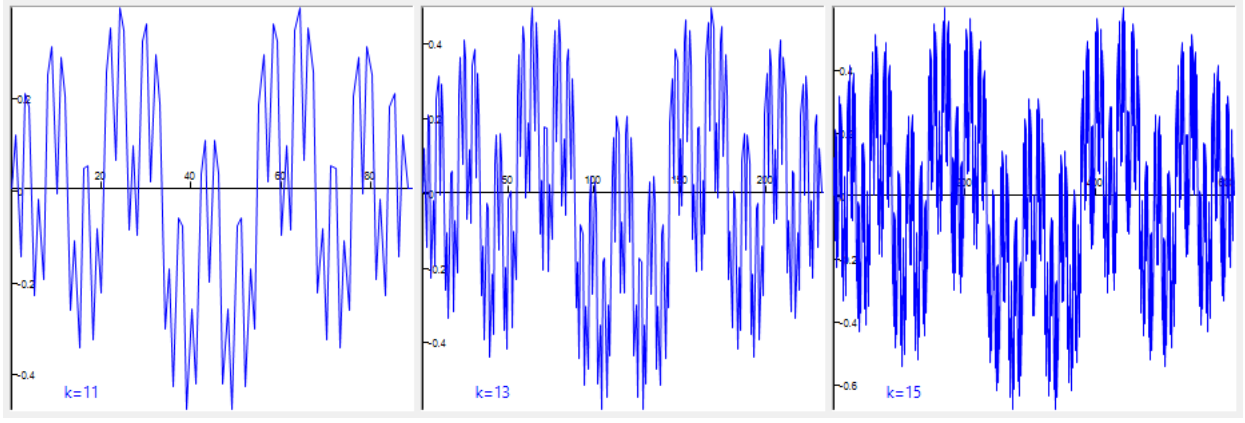


FIGURE 7.1. Renormalised graphs of (S_{kt}/ω^k) against t for $t = 0 \dots F_k$ at odd values of k . Note the slowly growing norm. We show below that the growth is $O(\log k)$.

For $1 \leq r \leq F_n - 1$ we now introduce a generalised $\xi_{\omega r}(\theta) = \{\theta + r\omega\} - 1/2$ so that our $\xi_{\infty t}$ of the previous section is now represented by $\xi_{\infty t}(0)$. We can now use the Fibonacci representation of t to split the sum S_{nt} into segments of length $b_s F_s$:

$$(7.15) \quad S_{nt} = \sum_{r=1}^t \sin \pi \omega^n \xi_{\omega s}(0) = \sum_{s=1}^m \sum_{r=1}^{b_s F_s} \sin \pi \omega^n \xi_{\omega r}(t_s \omega)$$

We now introduce a generalised $S_{nt}(\theta) = \sum_{r=1}^t \sin \pi (\omega^n \xi_{\omega r}(\theta))$ which allows us to write for $1 \leq t \leq F_n - 1$

$$(7.16) \quad S_{nt} = S_{nt}(0) = \sum_{s=1}^{m(t)} b_s S_{nF_s}(t_s \omega)$$

We proceed to study the order of the terms $S_{nF_s}(\theta)$.

Lemma 7.2. *Let p/q be a convergent of any real α . Then for any real θ*

$$\left| \sum_{i=1}^q \left(\{\theta + i\alpha\} - \frac{1}{2} \right) \right| < \frac{3}{2}$$

Proof. We can assume without loss of generality that $\alpha, \theta \in [0, 1)$. Since p/q is a convergent of α we have $\alpha - p/q = \nu/q^2$ for some $|\nu| < 1$.

Now $\theta = k/q + \phi$ for some $0 \leq k < q$ and $0 \leq \phi < 1/q$, and so $\theta + i\alpha = (k + ip)/q + \phi + i\nu/q^2$.

Suppose $\nu \geq 0$, then for $1 \leq i \leq q$ we have $(k + ip)/q \leq \theta + i\alpha < (k + ip + 2)/q$ and so $\{(k + ip)/q\} \leq \{\theta + i\alpha\}$ with the one exception that when $k + ip \equiv -1 \pmod{q}$ we may have $\phi + i\nu/q^2 \geq 1/q$ and then we can only write $\{(k + ip)/q\} - (q - 1)/q \leq \{\theta + i\alpha\}$.

Now $(p, q) = 1$, and so as i runs through $1, \dots, q$, $k + ip$ runs through a complete set of residues $0, \dots, q - 1 \pmod{q}$, and hence

$$(7.17) \quad \sum_{i=1}^q \{\theta + i\alpha\} \geq \left(\sum_{j=0}^{q-1} \frac{j}{q} \right) - \frac{q-1}{q} = \frac{1}{2}(q-1) - \frac{q-1}{q}$$

Similarly for $\nu < 0$ we have $(k + ip - 1)/q < \theta + i\alpha < (k + ip + 1)/q$ and so $\{(k + ip - 1)/q\} < \{\theta + i\alpha\}$ with the one exception that when $k + ip - 1 \equiv -1 \pmod{q}$ we may have $\phi + i\nu/q^2 \geq 0$ and then we can only write $\{(k + ip - 1)/q\} - (q - 1)/q \leq \{\theta + i\alpha\}$. Now summing as before also gives (7.17), and so this holds for any $|\nu| < 1$. We can immediately deduce

$$(7.18) \quad \sum_{i=1}^q \left(\{\theta + i\alpha\} - \frac{1}{2} \right) > -\frac{3}{2}$$

We now examine the upper bound of the sum. For $\nu \geq 0$ we have

$$(7.19) \quad \sum_{i=1}^q \{\theta + i\alpha\} \leq \sum_{j=0}^{q-1} \frac{j}{q} + \sum_{i=1}^q \left(\phi + \frac{i\nu}{q^2} \right) = \frac{1}{2}(q-1) + q\phi + \frac{1}{2}(q+1)\frac{\nu}{q} < \frac{1}{2}(q-1) + 1 + \frac{1}{2}\left(1 + \frac{1}{q}\right) = \frac{1}{2}q + 1 + \frac{1}{2q}$$

whilst for $\nu < 0$ we get

$$(7.20) \quad \sum_{i=1}^q \{\theta + i\alpha\} \leq \sum_{j=1}^q \frac{j}{q} + \sum_{i=1}^q \left(\frac{i\nu}{q^2} \right) = \frac{1}{2}(q+1) + \frac{1}{2}(q+1)\frac{\nu}{q} < \frac{1}{2}(q+1)$$

By adding $\sum_{i=1}^q (-1/2) = -(1/2)q$ to the two upper bound inequalities, the result follows by combining the three bounds obtained \square

We now fix n , and use the lemma to estimate S_{nF_i} for $1 \leq i < n$. We can do this by noting that F_{i-1}/F_i is a convergent to ω .

Using $\sin x = x + O(x^3)$ and from (4.7) $F_i\omega^{2n} \leq F_n\omega^{2n} = O(\omega^n)$, we can now apply the lemma to estimate $S_{nF_i}(\theta)$ as follows:

$$(7.21) \quad \begin{aligned} |S_{nF_i}(\theta)| &= \left| \sum_{p=1}^{F_i} \sin \pi \omega^n \left(\{\theta + p\omega\} - \frac{1}{2} \right) \right| \\ &= \pi \omega^n \left| \sum_{p=1}^{F_i} \left(\{\theta + p\omega\} - \frac{1}{2} + O(\omega^{2n}) \right) \right| \\ &< \pi \omega^n \left(\frac{3}{2} + O(\omega^n) \right) \end{aligned}$$

Note that the $O(\omega^n)$ term has no dependency on i .

We are now in a position to estimate S_{nt} for $1 \leq t \leq F_n - 1$, using (7.16) for $1 \leq t \leq F_n - 1$:

$$\begin{aligned} |S_{nt}(0)| &= \left| \sum_{s=1}^{m(t)} b_s S_{nF_s}(t_s \omega) \right| < \sum_{s=1}^m b_s \pi \omega^n \left(\frac{3}{2} + O(\omega^n) \right) \\ &< \pi \omega^n \left(\frac{3}{2} + O(\omega^n) \right) \sum_{s=1}^m b_s \end{aligned}$$

Now $\sum_{s=1}^m b_s$ is the Fibonacci length of t and by Lemma 4.7 $\sum_{s=1}^m b_s \leq \lfloor (\log t + 1) / \log(2 + \omega) \rfloor$ and so we have established:

Lemma 7.3. *For $1 \leq t \leq F_n - 1$, the partial sums $S_{nt} = \sum_{s=1}^t \sin \pi (\omega^n \xi_{\infty t})$ satisfy*

$$(7.22) \quad S_{nt} < \frac{3}{2} \pi \omega^n \lfloor (\log t + 1) / \log(2 + \omega) \rfloor + O(n\omega^{2n})$$

In particular we can find a K independent of n such that $|S_{nt}| < K\omega^n(\log t + 1)$

7.2.3. Conclusion of proof of convergence of the first sum. From Theorem 7.3 in section 7.2.2 we have $|S_{nt}| = |\sum_{s=1}^t \sin \pi(\omega^n \xi_{\omega s})| < K\omega^n(\log t + 1)$ for some K independent of n . Combining this with (7.14) we get $|C_{nt}S_{nt}| < \pi K(1 + O(\omega^{2n}))(\log t + 1)/t^2$. But $\sum(\log t + 1)/t^2$ is absolutely convergent, so putting $K_2 = \pi K \sum_{t=1}^{\infty} (\log t + 1)/t^2$ in (7.12) we get:

$$(7.23) \quad \lim_{n \rightarrow \infty} \sum_{t=1}^{M_n-1} \left| \left(\cot \left(\frac{\pi t}{F_n} \right) - \cot \left(\frac{\pi(t+1)}{F_n} \right) \right) \sum_{s=1}^t \sin \pi(\omega^n \xi_{\omega s}) \right| \leq K_2$$

So the sum above is absolutely convergent, and hence converges to a limit L_1^B . From (7.9) and (7.12) this gives us:

$$(7.24) \quad H_n \rightarrow H_n^* \longrightarrow L_1^B$$

7.3. Conclusion of proof of convergence of B_n . Combining (7.4), (7.6), (7.7) and (7.24) and gives us finally

$$(7.25) \quad \log B_n \rightarrow -2(L_1^B + L_2^B)$$

and noting that both limits are finite establishes Theorem 7.1.

8. TWO ADDITIONAL RESULTS

In this section we show how the other two results of Theorem 2.2 flow from our main result $P_{F_n}(\omega) \rightarrow c$. The first result is really just a direct corollary of our main result.

8.1. The convergence of $P_{F_n-1}(\omega)/F_n$.

Corollary 8.1. *The sequence $P_{F_n-1}(\omega)/F_n$ converges to $c\sqrt{5}/2\pi$ where c is the limit of the sequence $P_{F_n}(\omega)$*

Proof. Since $P_{F_n-1}(\omega) = P_{F_n}(\omega)/2 \sin \pi\omega^n \sim c/2\pi\omega^n$, the result follows from $F_n \sim \omega^{-n}/\sqrt{5}$ □

8.2. The power law growth of $P_k(\omega)$ for general k . We now turn to the more important result that the growth and decay of $P_k(\omega)$ is bounded by power laws, specifically:

Theorem 8.2. *There are real constants $K_1 \leq 0 < 1 \leq K_2$ independent of k such that for $k \geq 1$ we have $k^{K_1} \leq P_k(\omega) \leq k^{K_2}$*

The main part of the proof is to establish that these constants exist. If they do then our main result ($P_{F_n}(\omega) \rightarrow c$) shows we must have $K_1 \leq 0$, and Proposition 8.1 ($P_{F_n-1}(\omega)/F_n \rightarrow c\sqrt{5}/2\pi$) shows we must have $K_2 \geq 1$.

Knill and Tangeman provide an outline proof of existence in the logarithmic case, but appear to make an assumption which, although correct, seems to us to require its own proof. We will give the outline proof here, and then complete it rigorously.

Recall from section 4.4.1 that we can express any integer $k \geq 1$ as a sum of Fibonacci numbers $\sum_{s=1}^m b_s F_s$ subject to the rules ($b_s \in \{0, 1\}, b_m = 1, b_{r+1} = 1 \Rightarrow b_r = 0$). For $0 \leq s \leq m-1$ put $k_s = \sum_{u=s+1}^m b_u F_u$, $k_m = 0$ so that for $m > 1$ we can split the overall product into sub-products of length $b_s F_s$ (regarding the empty product as 1) to get:

$$\begin{aligned} P_k(\omega) &= \prod_{r=1}^k |2 \sin \pi(r\omega)| \\ &= \prod_{r=1}^{b_m F_m} |2 \sin \pi(r\omega)| \prod_{r=1}^{b_{m-1} F_{m-1}} |2 \sin \pi(r\omega + b_m F_m \omega)| \prod_{r=1}^{b_{m-2} F_{m-2}} |2 \sin \pi(r\omega + (b_m F_m + b_{m-1} F_{m-1})\omega)| \dots \\ (8.1) \quad &= \prod_{s=1}^m \prod_{r=1}^{b_s F_s} |2 \sin \pi(r\omega + k_s \omega)| \end{aligned}$$

Now the term for $s = m$ of this product is $\prod_{r=1}^{b_m F_m} |2 \sin \pi(r\omega)| = P_{F_m}(\omega) \sim c$ (by the main result of this paper), and it is also strictly positive, so that we can find constants $0 < C_1 < C_2$ bounding $P_{F_m}(\omega)$ for all m .

Conjecture 8.3. *Assume that we can choose real constants C_1, C_2 with $0 < C_1 < C_2$ such that they bound all the terms in (8.1), ie so that*

$$(8.2) \quad C_1 \leq \prod_{r=1}^{b_s F_s} |2 \sin \pi(r\omega + k_s \omega)| \leq C_2$$

for each $1 \leq s \leq m$. (Note that in order to bound empty products this requires $C_1 \leq 1 \leq C_2$).

Then we have from (8.1):

$$(8.3) \quad C_1^m \leq P_k(\omega) \leq C_2^m$$

Now for $m \geq 1$, using (4.7) we obtain $\log k \geq \log F_m = \log(\omega^{-m}/\sqrt{5})(1 - (-1)^m \omega^{2m}) > m \log \omega^{-1} - \log \sqrt{5} - 1$ which gives $m < (\log c'k)/\log \omega^{-1}$ for some constant c' . So for real but not necessarily positive constants K_1, K_2 :

$$(8.4) \quad k^{K_1} < P_k(\omega) < k^{K_2}$$

This is essentially an amplified version of the outline proof provided by Knill and Tangerman, although we have provided it in multiplicative form, rather than the additive (logarithmic) form used in the aforementioned paper. The assumption in conjecture 8.3 is in fact correct (and it is trivial if $b_s = 0$), but a proof does not appear trivial for $b_s = 1$, and so we provide one here.

Since the case $b_s = 0$ is trivial we need deal only with $b_s = 1$. By the rules of the Fibonacci decomposition (see Lemma 4.7), $b_r = 1$ implies $b_{r+1} = 0$. Hence $k_s \omega = \sum_{u=s+2}^m b_u F_u \omega = N + \sum_{u=s+2}^m -b_u (-\omega)^u$ for some integer N . Now $|\sum_{u=s+2}^m -b_u (-\omega)^u| \leq \omega^{s+2}(1 + \omega^2 + \omega^4 \dots) < \omega^{s+1}$. Hence the conjecture is proved if we can prove the slightly more general assertion:

Lemma 8.4. *There are real constants C_1, C_2 satisfying $0 < C_1 \leq 1 \leq C_2$ such that $C_1 \leq \prod_{r=1}^{F_n} |2 \sin \pi(r\omega + \alpha)| \leq C_2$ whenever $n \geq 2$ and $|\alpha| \leq \omega^{n+1}$.*

Note the lemma does not hold for $n = 1$ as $\prod_{r=1}^{F_n} |2 \sin \pi(r\omega + \alpha)| = 0$ for $\alpha = \omega^2$.

We begin by expanding the sine product as follows:

$$(8.5) \quad \prod_{r=1}^{F_n} |2 \sin \pi(r\omega + \alpha)| = \prod_{r=1}^{F_n} |2 \sin \pi(r\omega)| |\cos \pi\alpha + \cot \pi r\omega \cdot \sin \pi\alpha|$$

For $n \geq 2$ and $|\alpha| \leq \omega^{n+1}$ it is easy to calculate that $\cos \pi\alpha + \cot \pi r\omega \cdot \sin \pi\alpha > 0$. We can therefore take logs of the product above to obtain:

$$(8.6) \quad \log \prod_{r=1}^{F_n} |2 \sin \pi(r\omega + \alpha)| = \log P_{F_n}(\omega) + \sum_{r=1}^{F_n} \log \left(1 - 2 \sin^2 \frac{\pi\alpha}{2} + \cot \pi r\omega \cdot \sin \pi\alpha \right)$$

Since $P_{F_n}(\omega)$ is already suitably bounded by the main result of this paper, it remains to show that the log sum is bounded above and below.

We begin with establishing the upper bound as this is slightly more straightforward than the lower bound.

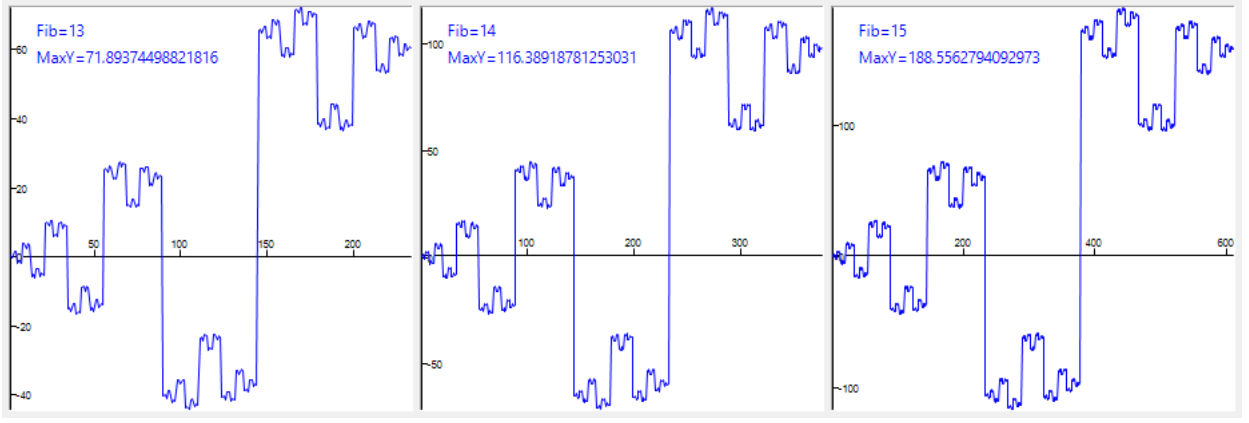


FIGURE 8.1. The renormalised graphs of $(-1)^n \sum_{r=1}^k \cot \pi r \omega$ over the Fibonacci interval $[1, F_n - 1]$ for $n = 13, 14, 15$. Note the remarkable scaling similarity.

8.2.1. *The upper bound on the growth rate.* We use $\log(1+x) \leq x$ for $x \in (-1, 1]$, and $\sin x > 2x/\pi$ from Lemma 4.2 to obtain:

$$\begin{aligned}
 \sum_{r=1}^{F_n} \log \left(1 - 2 \sin^2 \frac{\pi \alpha}{2} + \cot \pi r \omega \cdot \sin \pi \alpha \right) &< \sum_{r=1}^{F_n} \left(-2 \sin^2 \frac{\pi \alpha}{2} + \cot \pi r \omega \cdot \sin \pi \alpha \right) \\
 &= -2F_n \sin^2 \frac{\pi \alpha}{2} + \sin \pi \alpha \sum_{r=1}^{F_n} \cot \pi r \omega \\
 (8.7) \qquad &< \left(-2 \frac{\omega^{-n}}{\sqrt{5}} (1 + \omega^{2n}) \omega^{2n+2} \right) + \left(\pi \omega^{n+1} \left| \sum_{r=1}^{F_n} \cot \pi r \omega \right| \right)
 \end{aligned}$$

We now examine the sum $S_{F_n}(\omega) = \sum_{r=1}^{F_n} \cot \pi r \omega$. The sum $S_k(\omega) = \sum_{r=1}^k \cot \pi r \omega$ clearly has interesting relationships with our original product $P_k(\omega) = \prod_{r=1}^k |2 \sin \pi r \omega|$, and indeed it shows definite self-similar characteristics over Fibonacci intervals (see Figure 8.1) - for a detailed study of this sum see Knill [8]. For our current purposes we need to establish bounds on the growth, which we proceed to do as follows.

For $n \geq 3$, let $s(r) = [rF_{n-1}]$ so that as r runs through the values $1, \dots, F_n - 1$ so does s .

We consider n odd so that $\omega = (F_{n-1} + \omega^n)/F_n$. It follows that the fractional part of $r\omega$ lies in the interval $(s/F_n, (s+1)/F_n)$. Using $\cot x > \cot(x+\theta)$ when $x, x+\theta \in (0, \pi)$ and $\theta > 0$ gives us $\cot \pi s/F_n > \cot \pi r \omega$. We now use the fact that $\cot \pi x + \cot \pi(1-x) = 0$ to obtain:

$$(8.8) \qquad \sum_{r=1}^{F_n-1} \cot \pi r \omega < \sum_{r=1}^{F_n-1} \cot \pi \frac{s}{F_n} = 0$$

Similarly we have $\cot \pi r \omega > \cot \pi(s+1)/F_n$ but now $s+1$ runs through the values $2, \dots, F_n$. However the value of $s+1 = F_n$ results in a singularity of $\cot \pi$ so we treat separately the case $s = F_n - 1$. Let r^* be the value of r which satisfies $s(r^*) = F_n - 1$. Then from Lemma 4.4 we have $[F_{n-1}]^{-1} = [-F_{n-1}]$ for n

odd, and so $[r^*] = [F_{n-1}]^{-1}[F_n - 1] = [F_{n-1}]$, giving $r^* = F_{n-1}$. This gives us:

$$\begin{aligned}
 \sum_{r=1}^{F_n-1} \cot \pi r \omega &> \sum_{s=1}^{F_n-2} \cot \pi \frac{s+1}{F_n} + \cot \pi r^* \omega \\
 &= \left(\sum_{s=1}^{F_n-1} \cot \pi \frac{s}{F_n} - \cot \frac{\pi}{F_n} \right) + \cot (\pi F_{n-1} \omega) \\
 &= \left(0 - \cot \frac{\pi}{F_n} \right) - \cot \pi \omega^{n-1} \\
 (8.9) \quad &> -\frac{1}{\pi} (F_n + \omega^{-(n-1)}) \text{ for odd } n \geq 3
 \end{aligned}$$

But for n odd, $0 < \cot \pi F_n \omega = \cot \pi \omega^n < 1/\pi \omega^n$ and so we can add this to inequalities (8.8) and (8.9) to obtain for odd $n \geq 3$, that:

$$(8.10) \quad -\frac{1}{\pi} \left(\frac{1}{\sqrt{5}} (1 + \omega^{2n}) + \omega \right) < \omega^n \sum_{r=1}^{F_n} \cot \pi r \omega < \frac{1}{\pi}$$

By reversing signs appropriately, the same argument establishes an equivalent result for even $n \geq 4$, which in fact is easily verified to hold for $n = 2$:

$$(8.11) \quad \frac{1}{\pi} \left(\frac{1}{\sqrt{5}} (1 + \omega^{2n}) + \omega \right) > \omega^n \sum_{r=1}^{F_n} \cot \pi r \omega > \frac{-1}{\pi}$$

In both cases (even and odd) the left hand term is slightly larger in absolute value than the right. From (8.7), we therefore obtain for $n \geq 2$:

$$\begin{aligned}
 \sum_{r=1}^{F_n} \log \left(1 - 2 \sin^2 \frac{\pi \alpha}{2} + \cot \pi r \omega \cdot \sin \pi \alpha \right) &< \left(-2 \frac{\omega^{n+2}}{\sqrt{5}} (1 + \omega^{2n}) \right) + \omega \left(\frac{1}{\sqrt{5}} (1 + \omega^{2n}) + \omega \right) \\
 &< \omega \left(\frac{1}{\sqrt{5}} + \omega \right)
 \end{aligned}$$

This establishes the upper bound we needed, and also in (8.5) we now have for $n \geq 2$:

$$\prod_{r=1}^{F_n} |2 \sin \pi(r\omega + \alpha)| < P_{F_n}(\omega) e^{\omega \left(\frac{1}{\sqrt{5}} + \omega \right)}$$

This now also establishes the upper bound in (8.3) and hence also in (8.4). We now turn to the lower bound.

8.2.2. The lower bound on the growth rate. For the upper bound we were able to use the standard result that $\log(1+x) > x$. We now need a lower bound for the logarithm. The following lemma provides this:

Lemma 8.5. *For real $x > -0.683$ we have $\log(1+x) \geq x - x^2$*

Proof. For $x > -1$ put $f(x) = \log(1+x) - (x - x^2)$. Note the function is continuous on $(-1, \infty)$ and that $f(0) = 0$. It is easy to verify that this has critical points at $x = 0, -0.5$ and the derivative is positive on $(0, \infty)$ and negative on $(-0.5, 0)$ so that the function itself is positive on these two intervals. On $(-1, -0.5)$ the derivative is negative so the function descends with descending x from its maximum at $x = -0.5$ to a zero in $(-1, -0.5)$. A numerical calculation shows the root lies just below $x = -0.683$. \square

We wish to apply the lemma to the expression $\sum_{r=1}^{F_n} \log \left(1 - 2 \sin^2 \frac{\pi\alpha}{2} + \cot \pi r \omega \cdot \sin \pi\alpha \right)$ from (8.7). To do this we must first establish that $-2 \sin^2 \frac{\pi\alpha}{2} + \cot \pi r \omega \cdot \sin \pi\alpha > -0.683$. Now for $n \geq 4$ we have:

$$\begin{aligned}
\left| -2 \sin^2 \frac{\pi\alpha}{2} + \cot \pi r \omega \cdot \sin \pi\alpha \right| &< 2 \frac{\pi^2 \alpha^2}{4} + \pi \alpha \cot \pi \omega^n \\
&< \frac{\pi^2 \omega^{2n+2}}{2} + \pi \omega^{n+1} \frac{1}{\pi \omega^n (1 - \pi^2 \omega^{2n}/6)} \\
&< \frac{\pi^2 \omega^{10}}{2} + \frac{\omega}{(1 - \pi^2 \omega^8/6)} \\
&< 0.681
\end{aligned}$$

We can now apply the lemma to obtain:

$$\begin{aligned}
\sum_{r=1}^{F_n} \log \left(1 - 2 \sin^2 \frac{\pi\alpha}{2} + \cot \pi r \omega \cdot \sin \pi\alpha \right) &\geq \sum_{r=1}^{F_n} \left(-2 \sin^2 \frac{\pi\alpha}{2} + \cot \pi r \omega \cdot \sin \pi\alpha \right) \\
&\quad - \sum_{r=1}^{F_n} \left(-2 \sin^2 \frac{\pi\alpha}{2} + \cot \pi r \omega \cdot \sin \pi\alpha \right)^2 \\
&= \sum_{r=1}^{F_n} -2 \sin^2 \frac{\pi\alpha}{2} - 4 \sin^4 \frac{\pi\alpha}{2} \\
&\quad + \sum_{r=1}^{F_n} \left(1 + 4 \sin^2 \frac{\pi\alpha}{2} \right) \cot \pi r \omega \cdot \sin \pi\alpha - (\cot \pi r \omega \cdot \sin \pi\alpha)^2 \\
(8.12) \quad &\geq F_n \left(-2 \left(\frac{\pi \omega^{n+1}}{2} \right)^2 - 4 \left(\frac{\pi \omega^{n+1}}{2} \right)^4 \right) \\
&\quad - \left(1 + 4 \left(\frac{\pi \omega^{n+1}}{2} \right)^2 \right) \left| \sum_{r=1}^{F_n} \cot \pi r \omega \cdot \sin \pi\alpha \right| \\
&\quad - \sum_{r=1}^{F_n} (\cot \pi r \omega \cdot \sin \pi\alpha)^2
\end{aligned}$$

The first term is clearly bounded below. From (8.10), (8.11) and for $n \geq 2$, we have $|\sum \cot \pi r \omega \cdot \sin \pi\alpha| < \omega \left(\frac{1}{\sqrt{5}} (1 + \omega^{2n}) + \omega \right)$, and so the second term is also bounded below. It remains to show that the third term is bounded below. Using Lemma 4.2 (and allowing for $\alpha = 0$) we have for $n \geq 1$:

$$\sum_{r=1}^{F_n} (\cot \pi r \omega \cdot \sin \pi\alpha)^2 \leq (\pi\alpha)^2 \sum_{r=1}^{F_n} \cot^2 \pi r \omega$$

Using the same argument with $s(r) = [rF_{n-1}]$ as for the upper bound, we obtain for $n \geq 3$, $\cot^2 \pi r \omega < \cot^2 \pi s / F_n$ for $0 \leq s \leq [\frac{1}{2}F_n - 1]$ and for $n \geq 4$ it also gives $\cot^2 \pi r \omega < \cot^2(\pi(s+1)/F_n)$ for $[\frac{1}{2}F_n] \leq s \leq F_n - 1$. There is a special case: when $n \geq 4$ and F_n is odd there is an uncovered interval $[\frac{1}{2}(F_n - 1)/F_n, \frac{1}{2}(F_n + 1)/F_n]$, but here again $\cot^2 \pi r \omega < \cot^2 \pi s / F_n$ for $s = \frac{1}{2}(F_n - 1) = [\frac{1}{2}F_n]$. We are now almost ready to sum over r , but we again need to take care of singularities, and these occur this time at

$s = 0, F_n - 1$, corresponding to $r = F_n, [(-1)^n F_{n-1}]$. Hence for $n \geq 4$, using $|\cot x| < |1/x|$

$$\begin{aligned}
 \sum_{r=1}^{F_n} \cot^2 \pi r \omega &< \left(\sum_{s=1}^{\lfloor \frac{1}{2} F_n \rfloor} \cot^2 \pi \frac{s}{F_n} + \sum_{s=\lceil \frac{1}{2} F_n \rceil}^{F_n-2} \cot^2 \pi \frac{s+1}{F_n} \right) + \cot^2 \pi F_n \omega + \cot^2 \pi F_{n-1} \omega \\
 &< 2 \sum_{s=1}^{\lfloor \frac{1}{2} F_n \rfloor} \left(\frac{F_n}{\pi s} \right)^2 + \left(\frac{1}{\pi \omega^n} \right)^2 + \left(\frac{1}{\pi \omega^{n-1}} \right)^2 \\
 (8.13) \quad &< \frac{1}{\pi^2} F_n^2 \left(\frac{\pi^2}{6} \right) + \frac{1 + \omega^2}{\pi^2 \omega^{2n}}
 \end{aligned}$$

Hence for $n \geq 4$:

$$\begin{aligned}
 \sum_{r=1}^{F_n} (\cot \pi r \omega \cdot \sin \pi \alpha)^2 &< \pi^2 \omega^{2n+2} \left(\frac{1}{6} \frac{\omega^{-2n}}{5} (1 + \omega^{2n}) + \frac{1 + \omega^2}{\pi^2 \omega^{2n}} \right) \\
 &= \frac{\omega^2}{30} (\pi^2 (1 + \omega^8) + 30(1 + \omega^2))
 \end{aligned}$$

Hence the third term in (8.12) is also bounded below, and the lower bound we needed for this log sum is also established for $n \geq 4$. Hence for $n \geq 4$, (8.5) is bounded below by a strictly positive constant, and in fact it is easily verified that this is also true for $n = 2, 3$, finally establishing Lemma 8.4.

This now also establishes the lower bound in (8.3) and hence also in (8.4).

9. CONCLUSION

In this paper we studied Sudler's sine product in the important special case where ω is the golden ratio, thereby placing the work of Knill and Tangeman [6] on a rigorous footing. We now discuss directions for further research suggested by the work presented here and by the extensive discussion of open questions in [6].

In studies of quasi-periodic dynamics, it is usual to proceed from the golden mean, through quadratic irrationals to more general irrationals, sometimes with arithmetic conditions to overcome small-divisor obstructions. Sudler's sine product is well suited for such an approach. It is likely that the methods in sections 3–7 may be extended to all quadratic irrationals (with appropriate modification to take account of the (eventual) periodicity of the continued fraction expansion) and may provide a foundation to study the case of arbitrary irrational ω , leading to refinements of the norm and peak results presented in section 2.

Indeed, following Knill and Tangeman [6], we conjecture that for quadratic ω with a period ℓ continued fraction ($\ell \geq 1$) and with rational convergents p_k/q_k , the Sudler product for $n = q_k$ will converge to a periodic sequence of period dividing ℓ . Moreover, for ω satisfying other suitable arithmetic conditions such as Diophantine or Brjuno, the Sudler product will be bounded for $n = q_k$, where p_k/q_k are the rational convergents of ω . It is likely that a renormalisation approach will elucidate the overall structure of the Sudler product for arbitrary irrational ω .

Sudler's sine product appears in several areas of pure and applied mathematics. In the dynamical context, it arises in the renormalisation analysis of strange non-chaotic attractors for zero phase (see [5]). An analysis of non-zero phase leads to the more complex product

$$(9.1) \quad \prod_{r=1}^n 2 \sin \pi(r\omega + \alpha),$$

the analysis of which appears difficult in general, although we studied a special case (in which α decreases with n) in section 8 of this paper. Both the special case and the general case require obtaining the growth rate of the series

$$(9.2) \quad \sum_{r=1}^{F_n} \cot \pi r \omega .$$

We studied this growth rate in the case of the golden mean also in section 8. Again the results need to be generalised along the lines of the programme outlined above.

Coupled with the work of Knill and Lesieutre [9] (in which they used Herman's Denjoy-Koksma result [15] to study the generalised product

$$(9.3) \quad P_n(f, \omega) = \prod_{r=1}^n |f(r\omega)|$$

where $\log |f|$ is of bounded variation and ω is Diophantine), our results suggest that a fruitful research direction would be to adapt the methods in this paper to study in detail the product (9.3), first in the golden mean case and then for more general irrationals. It is likely that the symmetry properties of f will prove important in the application of the methods presented here.

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